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Automatic Numerical Generation of Body-Fitted Curvilinear Coordinate System for Field Containing Any Number of Arbitrary Two-Dimensional Bodies*

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A method for automatic numerical generation of a general curvilinear coordinate system with coordinate lines coincident with all boundaries of a general multi-connected region containing any number of arbitrarily shaped bodies is presented. No restrictions are placed on the shape of the boundaries, which may even be time-dependent, and the method is not restricted to two dimensions. With this procedure the numerical solution of a partial differential system may be done on a fixed rectangular field with a square mesh with no interpolation required regardless of the shape of the physical boundaries, regardless of the spacing of the curvilinear coordinate lines in the physical field, and regardless of the movement of the coordinate system. Numerical solutions for the lifting and nonlifting potential flow about Joukowski and Karman–Trefftz airfoils using this coordinate system generation show excellent comparison with the analytic solutions. The application to fields with multiple bodies is illustrated by a potential flow solution for multiple airfoils.

I. INTRODUCTION

There arises in all fields concerned with the numerical solution of partial differential equations the need for accurate numerical representation of boundary conditions. Such representation is best accomplished when the boundary is such that it is coincident with some coordinate line, for then the boundary may be made to pass through the points of a finite difference grid constructed on the coordinate lines. Finite difference expressions at, and adjacent to, the boundary may then be applied using only grid points on the intersections of coordinate lines without the need for any interpolation between points of the grid.

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The avoiding of interpolation is particularly important for boundaries with strong curvature or slope discontinuities, both of which are common in physical applications. Likewise, interpolation between grid points not coincident with the boundaries is particularly inaccurate with differential systems that produce large gradients in the vicinity of the boundaries, and the character of the solution may be significantly altered in such cases. In many differential systems the boundary conditions are the dominant influence on the character of the solution, and the use of grid points not coincident with the boundaries thus places the most inaccurate difference representation in precisely the region of greatest sensitivity. The generation of a curvilinear coordinate system with coordinate lines coincident with all boundaries (herein called a "natural coordinate system" for purposes of identification) is thus an essential part of a numerical solution.

A general method of generating natural coordinate systems is to let the natural coordinates be solutions of an elliptic partial differential system in the physical plane, with Dirichlet boundary conditions on all boundaries. The procedure is not restricted to two dimensions, allows the coordinate tangential to the boundary to be distributed quite easily as desired along the boundary, and is applicable to all multiconnected regions (and thus to the flow about any number of arbitrarily shaped bodies). The coordinate system so generated is not necessarily orthogonal, but orthogonality is not required, and its lack only requires that the partial differential system to be solved on the coordinate system when generated must be transformed directly through implicit partial differentiation rather than by use of the scale factors and differential operators developed for orthogonal curvilinear systems. An orthogonal system cannot be achieved with arbitrary spacing of the natural coordinate lines around the boundary, and the capability for such arbitrary spacing is of more importance than orthogonality.

This idea has been applied previously to two-dimensional regions interior to a closed boundary (simply connected regions) by Winslow [1], Barfield [2], Chu [3], Amsden and Hirt [4], and Godunov and Prokopov [5]. Winslow [1] and Chu [3] take the natural coordinates to be solutions of Laplace's equation in the physical plane which, as is shown in the next section, makes the physical cartesian coordinates solutions of a quasi-linear elliptic system in the transformed plane. Barfield [2] and Amsden and Hirt [4] reverse the procedure, taking the physical coordinates to be solutions in the transformed plane of a linear elliptic system which consists of Laplace's equation modified by a multiplicative constant on one term. This makes the natural coordinates solutions of a quasi-linear elliptic system in the physical plane. (Barfield also considers a hyperbolic system, but such a system cannot be used to treat closed boundaries, since only elliptic systems allow specification of boundary conditions on the entirety of closed boundaries.)

Amsden and Hirt [4] construct the coordinate generation method by iterative weighted averaging of the values of the physical coordinates at fixed points in the transformed plane in terms of values at neighboring points. Although not stated as such, this procedure is precisely equivalent to solving Laplace's equation, or a modification thereof of the form noted above in Barfield [2], for the physical coordinates in the transformed plane by Gauss–Seidel iteration. (Amsden and Hirt also allow the boundary to move at each iteration, but this is simply equivalent to approaching the solution of the boundary-value problem through a succession of boundary-value problems converging to the problem of interest.) In the approach of Godunov and Prokopov [5] the elliptic system is quasi-linear in both the physical and transformed plane. (These authors apply a second transformation to that used by Chu [3], the transformation functions of this latter transformation being chosen a priori to control the coordinate spacing. Though not stated as such, the overall transformation may be shown to be generated by taking the transformed coordinates to be solutions in the physical plane of Laplace's equation modified by the addition of a multiple of the square of the Jacobian, the multiplicative factors being a priori chosen functions of the physical coordinates.)

In the present research this technique of generating the natural coordinates as solutions of an elliptic differential system in the physical plane has been extended to multiconnected regions with any number of arbitrarily shaped bodies or holes. Present effort is confined to two dimensions in the interest of computer economy, but all techniques are immediately extendable to three dimensions. The transformation procedure is explained, illustrated, and confirmed by application to the solution for potential flow about several bodies in the following sections.

II. Basic Transformation Method

A. *Mathematical Construction*

Let it be desired to transform the two-dimensional, doubly connected region, R, bounded by two closed contours of arbitrary shape into a rectangular region, R', as shown in Fig. 1. The general transformation from the physical plane (x, y) to the transformed plane (ξ, η) is given by $\xi = \xi(x, y)$, $\eta = \eta(x, y)$. Similarly, the inverse transformation is given by $x = x(\xi, \eta)$, $y = y(\xi, \eta)$. Derivatives are transformed as follows:

$$f_x = \frac{\partial(f, y)}{\partial(\xi, \eta)} \Big/ \frac{\partial(x, y)}{\partial(\xi, \eta)} = \frac{(y_\eta f_\xi - y_\xi f_\eta)}{J} \tag{1a}$$

$$f_y = \frac{\partial(x, f)}{\partial(\xi, \eta)} \Big/ \frac{\partial(x, y)}{\partial(\xi, \eta)} = \frac{(-x_\eta f_\xi + x_\xi f_\eta)}{J} \tag{1b}$$

where J is the Jacobian of the transformation, $J = x_\xi y_\eta - x_\eta y_\xi$.

Since the basic idea of the transformation is to generate transformation functions

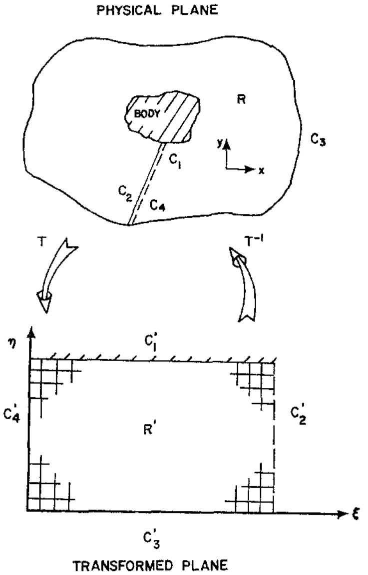

FIG. 1. Field transformation–single body.

such that all boundaries are coincident with coordinate lines, the natural coordinates (ξ, η) are taken as solutions of some suitable elliptic boundary value problem with one of these coordinates constant on the boundaries. Using Laplace's equation as the generating elliptic system, we have

$$\xi_{xx} + \xi_{yy} = 0 \tag{2a}$$

$$\eta_{xx} + \eta_{yy} = 0 \tag{2b}$$

with Dirichlet boundary conditions, η = constant = η_1 on C_1, η = constant = η_2 on C_3; $\xi(x, y)$ a multiple valued solution with a branch of $\xi(x, y)$ specified (but not constant) on C_1 and C_3. The curve C_1 on the physical plane transforms to the upper boundary, C_1', of the transformed plane. Similarly, C_3 transforms to C_3', etc. The right and left boundaries of the rectangular transformed plane, C_2' and C_4', are coincident in the physical plane. The curve which transforms to these boundaries connects C_1 and C_3 and determines a branch cut for the multiple valued function $\xi(x, y)$. Thus the functions and all derivatives are continuous across this cut.

Now since we wish to do all numerical computation in the rectangular transformed plane, it is necessary to interchange the dependent and independent variables in (2). Thus

$$\alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} = 0 \tag{3a}$$

$$\alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} = 0 \tag{3b}$$

where

$$\alpha = x_\eta^2 + y_\eta^2 \tag{3c}$$

$$\beta = x_\xi x_\eta + y_\xi y_\eta \tag{3d}$$

$$\gamma = x_\xi^2 + y_\xi^2 \tag{3e}$$

with the transformed boundary conditions, $x = f_1(\xi, \eta_1)$ on C_1', $y = g_1(\xi, \eta_1)$ on C_1', $x = f_2(\xi, \eta_2)$ on C_3', $y = g_2(\xi, \eta_2)$ on C_3'.

This system is a quasi-linear elliptic system with Dirichlet boundary conditions for the physical coordinates in the transformed plane. The differential equations of the system (3) are considerably more complicated than those of Eq. (2). However, the boundary conditions of Eq. (3) are specified on straight boundaries, and the computation field is rectangular. We have thus exchanged a problem having simple equations but complex boundary conditions for a problem having complex equations and simple boundary conditions. This statement also holds for all partial differential equations solved on the natural coordinates.

The natural coordinate system so generated has a constant η-line coincident with each boundary in the physical plane. The ξ-lines may be spaced in any manner desired around the boundaries by specification of the ξ boundary conditions, or equivalently by specification of (x, y) at the equi-spaced ξ-points on the η_1 and η_2 lines of the transformed plane. Control of the spacing of the η-lines may be exercised by varying the elliptic system of which ξ and η are solutions. For example, the use of the system (2) with equi-spaced points on circular boundaries will produce an expanding cylindrical coordinate system, while the addition of $1/\eta$ to the right side of (2b) will produce the common system of cylindrical coordinates.

As noted above, orthogonality is not required of a coordinate system for solution of a system of partial differential equations. Indeed, normal derivatives on the boundaries may be easily represented in a non-orthogonal system as follows: Let $y = g(x)$, $(y - g(x) \equiv f(x, y) = 0)$ be the equation of some boundary in cartesian coordinates. Then

$$\begin{aligned}\frac{\partial f}{\partial n} &= \mathbf{n} \cdot \nabla f = \frac{\mathbf{i}g' - \mathbf{j}}{((g')^2 + 1)^{1/2}} \cdot (\mathbf{i}f_x + \mathbf{j}f_y) \\ &= \frac{1}{J(1 + (g')^2)^{1/2}} [f_\xi(g'y_\eta + x_\eta) - f_\eta(g'y_\xi + x_\xi)] \end{aligned} \tag{4}$$

where $g' \equiv dy/dx$. All derivatives in the last expression can be calculated along coordinate lines in the natural coordinate system. Thus it is not necessary to require orthogonality at the body surface in order to get an accurate representation of the normal derivative.

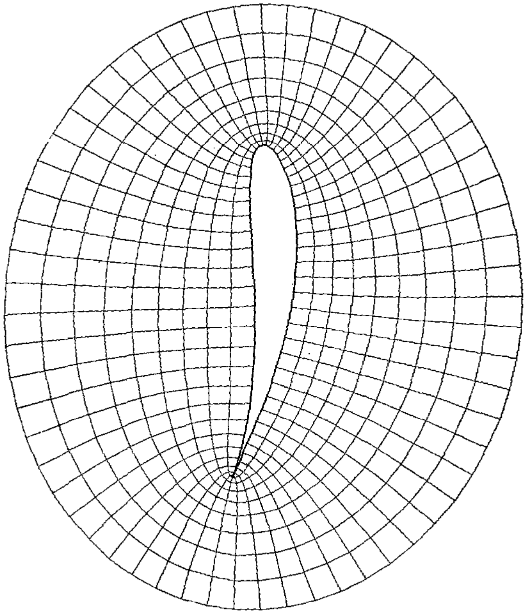

FIG. 2. Coordinate system–Joukowski airfoil.

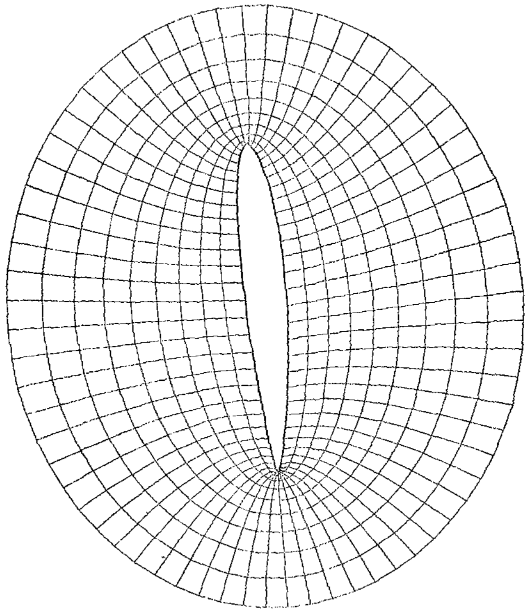

FIG. 3. Coordinate system–Karman–Trefftz airfoil.

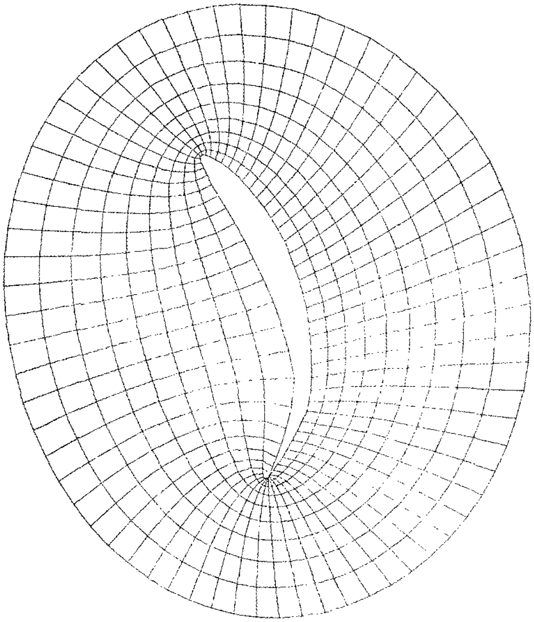

FIG. 4. Coordinate system–flapped Karman–Trefftz airfoil.

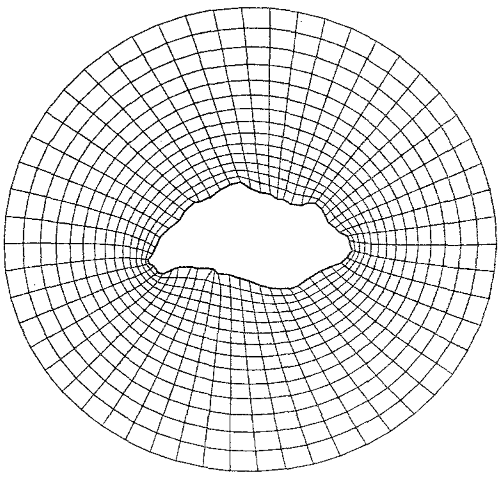

FIG. 5. Coordinate system–rock.

Once the (ξ, η) system is obtained by solving Eq. (3) we may then solve any set of partial differential equations on this natural coordinate system by solving the transformed equations on the rectangular transformed field. Thus regardless of the shape of the body and regardless of the spacing of the natural coordinates, all numerical computations, both to generate the coordinate system and to subsequently solve the partial differential system of interest, are done on a rectangular grid with a square mesh, i.e., in the transformed plane. Finally, it is shown in the Appendix that physical integral conservation relations need not be lost in the transformed plane.

B. *Numerical Solution of the Transformation Equations*

The set (3) has been successfully solved for a number of bodies—Joukowski and Karman–Trefftz airfoils and an arbitrarily shaped rock—(Figs. 2–5) using accelerated Gauss–Seidel iteration. The plots given are of $\xi =$ constant lines and $\eta =$ constant lines in the physical plane. An analytical solution of Eq. (3) can be obtained for circular boundaries with equispaced points and was used as a test case for the computer routine.

C. *Application to Potential Flow*

In order to verify the usefulness of the natural coordinate systems described above, the Laplace equation for the stream function for lifting and non-lifting potential flow was solved numerically using the curvilinear coordinate system. When transformed this equation becomes

$$\alpha\psi_{\xi\xi} - 2\beta\psi_{\xi\eta} + \gamma\psi_{\eta\eta} = 0 \tag{5}$$

where α, β, and γ are given by (3c, d, e), and the transformed boundary conditions are

$$\psi(\xi, \eta) = 0 \text{ on } \eta = \eta_1 \text{ (i.e., on } C_1)$$

$$\psi(\xi, \eta) = U_\infty y(\xi, \eta_2) \cos\theta - U_\infty x(\xi, \eta_2) \sin\theta \text{ on } \eta = \eta_2 \text{ (i.e., on } C_3)$$

where θ is the angle of attack. The uniqueness is implied by insisting that the solution be periodic in $-\infty < \xi < \infty$, $\eta_1 \leqslant \eta \leqslant \eta_2$. α, β, and γ are calculated during the generating of the natural coordinate system. Equation (5) was approximated using second-order, central differences for all derivatives, and the resulting difference equation was solved by accelerated Gauss-Seidel iteration on the rectangular transformed field.

Plots of the stream function contours for the airfoils and the rock are given in Figs. 6–10. The comparison with the analytic solution[6] for the Joukowski and

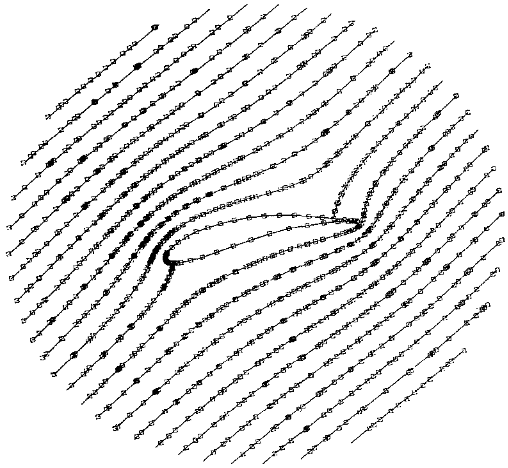

FIG. 6. Nonlifting potential flow–Joukowski airfoil.

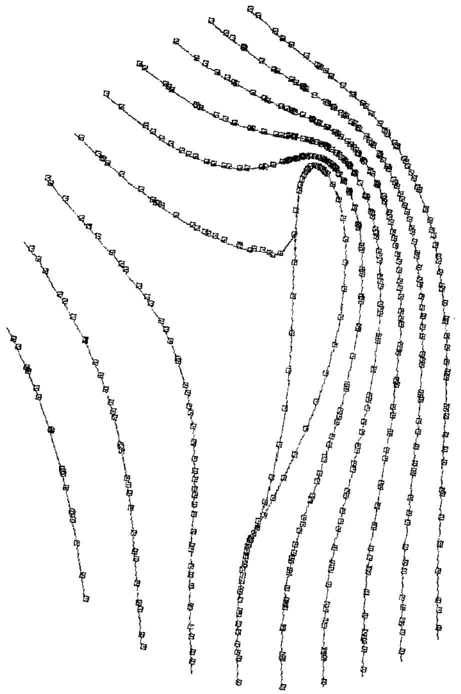

FIG. 7. Lifting potential flow–Joukowski airfoil.

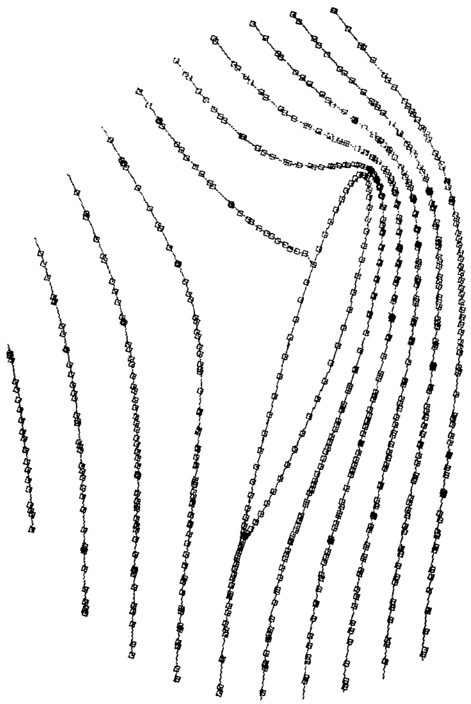

FIG. 8. Lifting potential flow–Karman–Trefftz airfoil.

Karman–Trefftz airfoils is excellent. In particular, it should be noted that the points of intersection of the zero streamline with the airfoils agree with the analytic solutions. These points were not controlled in any manner by the coordinate system (Figs. 2–4) or by any other means, but were entirely free to be determined by the numerical solution. (The solid lines represent the analytic solution in Figs. 6–9, and the squares the numerical solution. The plots are unretouched contour plots produced on a Gould Electrostatic Plotter. Only a portion of the computational field used is shown in these figures.) The excellent correspondence between the numerical and analytic results on these figures reflects the fact that the rms error between the numerical and analytic solutions was only 0.001 % of the maximum value of the stream function on the field. It is clear from these results that the use of body-fitted curvilinear system can lead to highly accurate numerical solutions.

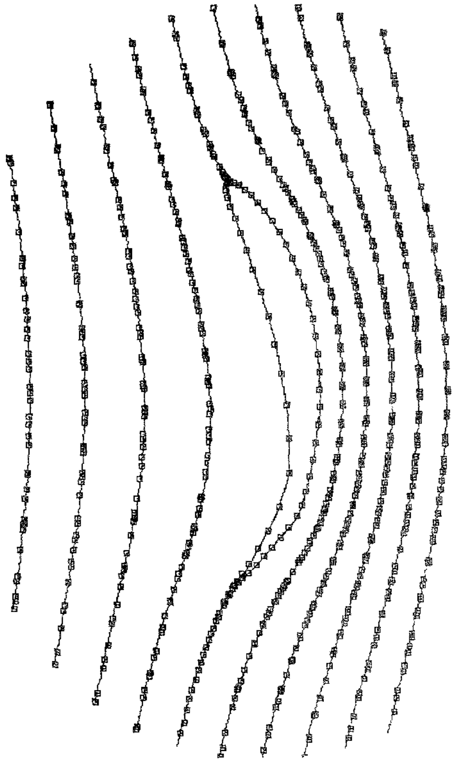

FIG. 9. Lifting potential flow–flapped Karman–Trefftz airfoil.

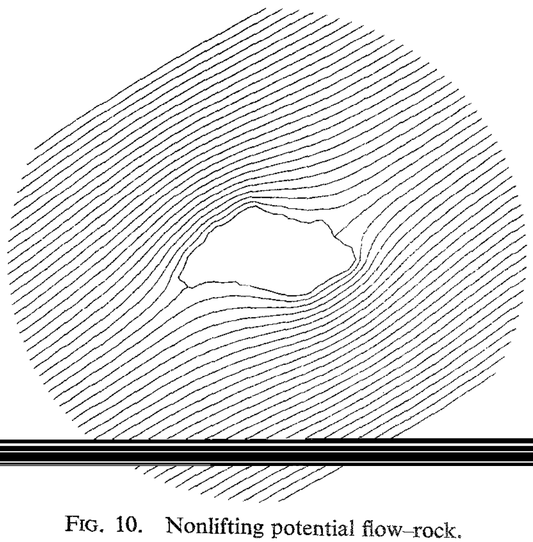

FIG. 10. Nonlifting potential flow–rock.

III. Extensions

A. *Regions With Multiple Bodies*

The same procedure for natural coordinate generation may be extended to regions that are more than doubly connected, i.e., have more than two closed boundaries or, equivalently, more than one body or hole within a single outer boundary. The transformation to the rectangular field is illustrated in Fig. 11.

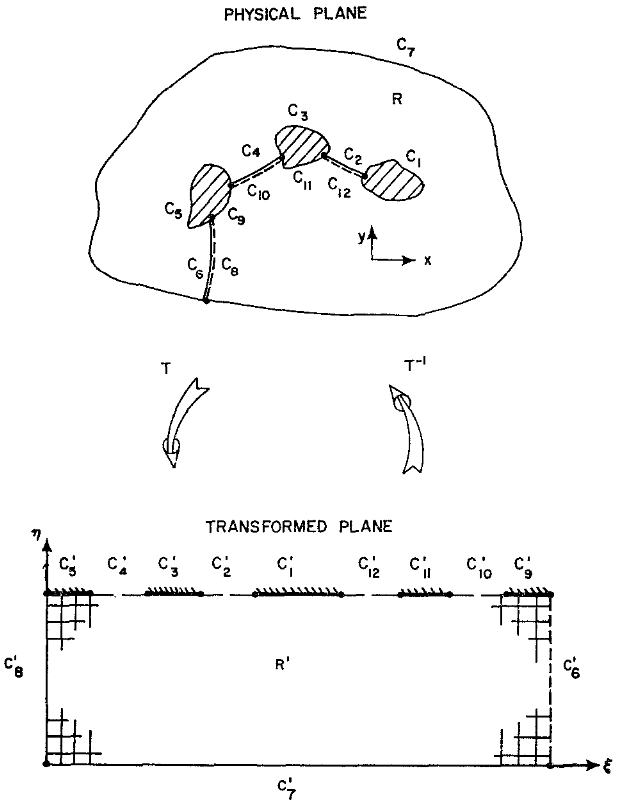

Fig. 11. Field transformation–multiple bodies.

We require that the η-coordinate be equal to the same constant on all the interior boundaries, i.e., on all "bodies" in the field. Let all the bodies be connected by arbitrary cuts and, similarly, one body be connected to the outer boundary by an arbitrary cut. Since the η-coordinate is equal to the same constant on all the bodies, it is, of course, equal to that constant on the cuts between the bodies also. By contrast, the ξ-coordinate is taken constant on the cut between the bodies and the outer boundary. Since the locations of these cuts are not specified, the specification of η or ξ as constant on a cut does not overspecify the elliptic problem.

Note that all bodies except one are split into two segments. Each cut appears twice on the transformed field boundary, of course, the two segments corresponding to the two "sides" of the cut in the physical plane and thus being re-entrant boundaries with the functions and all derivatives continuous thereon. We thus have (x, y) specified on the portions of the upper boundary of the transformed field that correspond to the bodies, and also on the entire lower boundary, corresponding to the outer boundary in the physical field. The remaining portions of the upper boundary and the entire side boundaries are re-entrant boundaries, and thus neither require nor allow specification of (x, y) thereon.

Again an elliptic Dirichlet problem is solved to generate the natural coordinates (ξ, η), as in the previously-considered case with only a single body. All computation, both to generate the coordinates and subsequently to solve the partial differential system of interest, are again done on the rectangular field with square mesh in the transformed plane. Figures 12 and 13 show the coordinate system and lifting

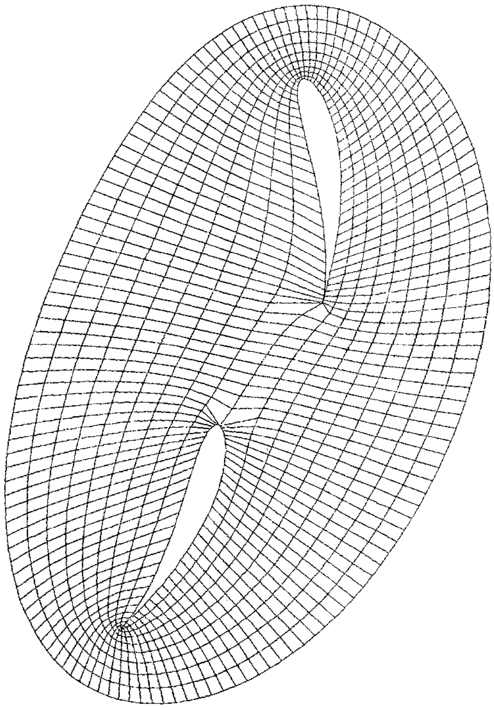

FIG. 12. Coordinate system–multiple airfoils.

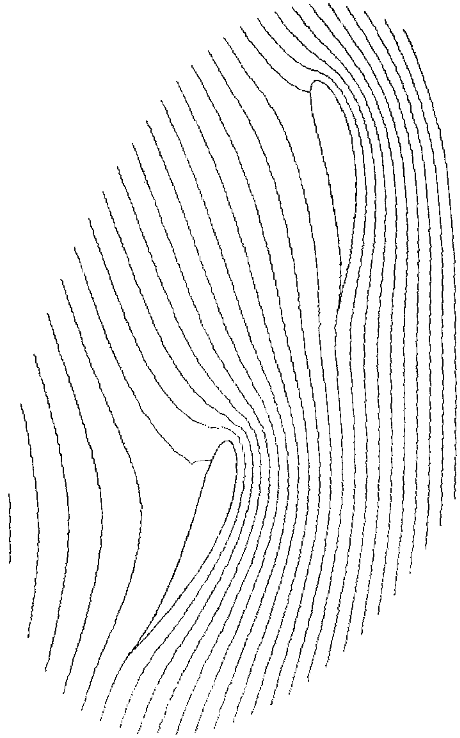

FIG. 13. Lifting potential flow–multiple airfoils.

potential flow for multiple airfoils. Note that there are no discontinuities in the streamlines across the cut between the bodies.

B. *Time-Dependent Coordinate Systems*

Now suppose the coordinate system changes with time, i.e., the grid points move in the physical plane. Ordinarily such movement of the physical grid points would require interpolation among the grid points to produce values of the dependent variables at the new locations of the grid points. With the present method of coordinate system generation, however, it is possible to perform all computation on the fixed rectangular grid in the transformed plane *without any interpolation* no matter how the grid points move in the physical plane as time progresses. This occurs as follows:

Recall that the natural coordinate system is generated as the solution of some elliptic system with the values of the transformed coordinates (ξ, η) specified on the boundaries in the physical plane, one of these coordinates being specified to be constant on the boundaries and the other being distributed as desired along the boundaries in order, perhaps, to concentrate grid points in certain regions. The transformed coordinates define a rectangular plane, the extent of which is determined by the range of the values of ξ and η. Now if the same boundary values of ξ and η are redistributed in the physical plane, perhaps because the boundaries in the physical plane have actually moved or maybe just to change the concentration of grid points around the boundaries, and a suitable elliptic system is solved for the transformed coordinates with these new boundary conditions, new transformation functions can be produced with still the same range of values in ξ and η (provided the elliptic system used exhibits a maximum principle) and hence to the same rectangular field in the transformed plane. The grid points in the rectangular transformed plane thus remain stationary, and the effect of the movement of the coordinate system in the physical plane is then just to change the values of the physical coordinates (x, y) at the fixed grid points in the rectangular transformed plane.

Thus, although the position of a grid point changes on the physical plane, its position in the transformed plane is fixed. Also, with the time derivative transformed to the transformed plane as shown below:

$$\begin{aligned}\left(\frac{\partial f}{\partial t}\right)_{x,y} = \frac{\partial(x, y, f)}{\partial(\xi, \eta, t)} \bigg/ \frac{\partial(x, y, t)}{\partial(\xi, \eta, t)} = \left(\frac{\partial f}{\partial t}\right)_{\xi,\eta} &- \frac{1}{J}\left(\frac{\partial f}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial f}{\partial \eta}\frac{\partial y}{\partial \xi}\right)\left(\frac{\partial x}{\partial t}\right)_{\xi,\eta} \\ &+ \frac{1}{J}\left(\frac{\partial f}{\partial \xi}\frac{\partial x}{\partial \eta} - \frac{\partial f}{\partial \eta}\frac{\partial x}{\partial \xi}\right)\left(\frac{\partial y}{\partial t}\right)_{\xi,\eta}\end{aligned} \tag{6}$$

all derivatives are expressed in the transformed plane, so that the interpolation that would be necessary to supply values at grid points in the physical plane that have moved is not required in the transformed plane. (Note that in the transformed expression for the time derivative, all derivatives are taken at the fixed grid points in the transformed plane. The movement of the grid in the physical plane is reflected only through the rates of change of x and y at the fixed grid points in the transformed plane.)

It is thus possible to cause the natural coordinate system to change in time however desired and still have all computation done on a fixed rectangular grid with square mesh without the need of any interpolation. This allows the natural coordinate system in the physical plane to deform with a deforming body, blast front, shock, free surface, or any other boundary, keeping a coordinate line always coincident with the boundary at all times. It also allows the concentration of mesh points to be changed as desired with time.

Mathematically what has transpired is that the original problem, consisting say

boundary conditions specified on moving general boundaries, has been transformed to a system of $N + 2$ partial differential equations (the two elliptic equations for the natural coordinates having been added) with boundary conditions that are now time-dependent but specified on steady rectangular boundaries. The physical coordinate system has thus been, in effect, eliminated from the problem, at the expense of adding two elliptic equations to the original system.

Moving boundaries have been treated in simply-connected regions in the work of Amsden and Hirt [4] and Godunov and Prokopov [5].

C. *Control of Spacing of Coordinate Lines*

As noted above, the spacing of the coordinate lines emanating from a boundary is easily controlled at the boundary, since values of this coordinate are specified as desired on the boundary. Control of the spacing of coordinate lines in the field, however, must be accomplished by altering the elliptic system that is solved to generate the coordinates. Only elliptic systems that exhibit a maximum principle are acceptable, however, since the entire bounded physical region, R, must map onto the rectangular region, R', in the transformed plane. A maximum principle guarantees that the maximum values of the curvilinear coordinates occur on the boundary of the physical region, R, so that the mapping is as required. This aspect of the method of natural coordinate generation is one which will have to be studied in some detail before optimum methods of coordinate line spacing control evolve. However, one method of control is as follows:

One other elliptic equation for which the maximum principle does hold is the equation

$$\nabla^2 f = P(f - f_0)$$

where P is a continuously differentiable positive function of x and y and f_0 is a constant between the maximum and minimum boundary values of f. A natural coordinate system may be generated by solving the two boundary value problems consisting of the equations

$$\nabla^2 \xi = P(\xi - \xi_2)$$
$$\nabla^2 \eta = Q(\eta - \eta_2)$$

with the boundary conditions given above. The functions P and Q are continuously differentiable positive functions on R and its boundary and the constants ξ_2 and η_2 are the minimum boundary values of $\xi(x, y)$ and $\eta(x, y)$, respectively. The effect of replacing Laplace's equation by these elliptic equations is the following.

Let ξ_h and η_h be the harmonic functions with the same boundary values. Since $\nabla^2 \xi \geqslant 0$ and $\nabla^2 \eta \geqslant 0$, the solutions ξ and η of the boundary value problems are subharmonic on R. A basic property of subharmonic functions gives the inequalities $\xi \leqslant \xi_h$ and $\eta \leqslant \eta_h$ throughout the region R. Thus replacing $\nabla^2 \xi = 0$ by

$\nabla^2\xi = P(\xi - \xi_2)$ causes the interior of the ξ = constant coordinate lines in the physical plane to move in the clockwise direction. This has the effect of decreasing the positive angle from the contour C_1 to the coordinate line. Replacing $\nabla^2\eta = 0$ by $\nabla^2\eta = Q(\eta - \eta_2)$ causes the η = constant coordinate lines to move closer to the contour C_1, hence, resulting in a more expanding system.

Instead of the equations above, it might be more desirable in certain problems to use either $\nabla^2\xi = P(\xi - \xi_1)$ or $\nabla^2\eta = Q(\eta - \eta_1)$ in place of Laplace's equation, where ξ_1 and η_1 are the maximum boundary values of ξ and η. These equations would produce the opposite effect on the natural coordinate system.

In regard to fluid mechanics computations, the intriging possibility exists of taking P and Q to be dependent on the vorticity magnitude, or other gradients, and thus causing the coordinate lines to concentrate automatically in regions of high gradients in the flow field, allowing the coordinate system to be time-dependent as discussed in the preceding section. This might also be a way of handling shocks automatically, without smearing or marching procedures, if the coordinate lines concentrate automatically in regions of large gradients.

Both Barfield [2] and Amsden and Hirt [4] achieve some control of the coordinate line spacing by varying the elliptic system as follows: As noted previously, these methods take the physical coordinates to be solutions of modified Laplace equations in the transformed plane (and hence the curvilinear coordinates to be solutions of a nonlinear elliptic system in the physical plane):

$$x_{\xi\xi} + a x_{\eta\eta} = 0$$
$$y_{\xi\xi} + b y_{\eta\eta} = 0$$

Some control of the coordinate line spacing in the field may then be exercised by varying the constants a and b. Both investigators reported some crossover of coordinate lines for some values of a and b or for some distributions of x and y on the boundaries.

Amsden and Hirt also used different weights on each neighboring value in their weighted-average iteration for the x and y values in the field in the transformed plane, again in an attempt to control the coordinate line spacing. This is equivalent to solving a nonlinear elliptic system for the physical coordinates in the transformed plane.

To achieve some control of the spacing, Godunov and Prokopov [5] added the terms $ax_\xi + bx_\eta$ and $ay_\xi + by_\eta$, where $a = a(\xi, \eta)$ and $b = b(\xi, \eta)$ are specified functions, to the right sides of Eqs. (3a) and (3b), respectively. These equations may be shown to be the transformation of the following equations in the physical plane:

$$\xi_{xx} + \xi_{yy} = -a(\xi_x\eta_y - \xi_y\eta_x)^2$$
$$\eta_{xx} + \eta_{yy} = -b(\xi_x\eta_y - \xi_y\eta_x)^2.$$

IV. Conclusion

The general method of boundary-fitted coordinate system generation discussed should have wide applicability in the numerical solution of partial differential equations. With its use, the treatment of fields with complex boundaries and any number of bodies or holes is not inherently more difficult than problems with simply geometry. The method affords a natural means of treating problems with moving or deforming boundaries, since the computational field remains steady in any case and no interpolation is required. Finally, the complete coupling of the partial differential equations for the coordinate system with those of the physical problem of interst, so that the coordinate system as such is effectively eliminated, is an intriging area for further pursuit.

Appendix

It should be noted that the divergence property is preserved in the transformed plane. For example, consider the partial differential equation

$$\nabla \cdot \mathbf{F} = S \tag{A1}$$

If this equation is integrated over an area, R, in the two-dimensional physical plane we have

$$\iint_R (F_{1_x} + F_{2_y})\, dx\, dy = \iint_R S\, dx\, dy$$

where the x and y components of $\mathbf{F}$ are indicated by subscripts 1 and 2, respectively, to avoid confusion with the notation for partial deviatives. Application of Green's Theorem to the integral on the left yields the conservation relation

$$\oint_C (F_1\, dy - F_2\, dx) = \iint_R S\, dx\, dy \tag{A2}$$

where C is the boundary curve of the area R. Here the contour integral represents a net flux through the closed boundary of the area R, and the integral over the area represents a source within the area.

Now let the partial differential equation be transformed so that in the transformed plane we have

$$\frac{1}{J}(F_{1_\xi} y_\eta - F_{1_\eta} y_\xi) + \frac{1}{J}(F_{2_\eta} x_\xi - F_{2_\xi} x_\eta) = S \tag{A3}$$

But note that

$$(F_1 y_\eta - F_2 x_\eta)_\xi + (F_2 x_\xi - F_1 y_\xi)_\eta = (F_{1_\xi} y_\eta - F_{2_\xi} x_\eta) + (F_{2_\eta} x_\xi - F_{1_\eta} y_\xi)$$
$$+ (F_1 y_{\eta\xi} - F_2 x_{\eta\xi}) + (F_2 x_{\xi\eta} - F_1 y_{\xi\eta})$$

and the last two terms cancel. Therefore the partial differential equation in the transformed plane may be equivalently written as

$$(F_1 y_\eta - F_2 x_\eta)_\xi + (F_2 x_\xi - F_1 y_\xi)_\eta = SJ \tag{A4}$$

Now integrate this equation over the transformed area R':

$$\iint_{R'} [(F_1 y_\eta - F_2 x_\eta)_\xi + (F_2 x_\xi - F_1 y_\xi)_\eta] \, d\xi \, d\eta = \iint_{R'} SJ \, d\xi \, d\eta$$

Again using Green's Theorem for the integral on the left, we have the conservation relation in the transformed plane:

$$\oint_{C'} [(F_1 y_\eta - F_2 x_\eta) \, d\eta - (F_2 x_\xi - F_1 y_\xi) \, d\xi] = \iint_{R'} SJ \, d\xi \, d\eta \tag{A5}$$

Again the contour integral represents a net flux through the boundary curve, C', as may be seen in the following. The unit normal to a ξ-coordinate line is given by

$$\mathbf{n}^{(\xi)} = \frac{d\mathbf{r}}{ds} x \mathbf{k} \Big/ \left| \frac{d\mathbf{r}}{ds} \right|$$

[illegible]
along the line, and $\mathbf{k}$ is the unit normal to the two-dimensional plane. Now

$$\frac{d\mathbf{r}}{ds} = \mathbf{i} \frac{dx}{ds} + \mathbf{j} \frac{dy}{ds} = (\mathbf{i} x_\eta + \mathbf{j} y_\eta) \frac{d\eta}{ds}$$

so that

$$\mathbf{n}^{(\xi)} = (\mathbf{i} y_\eta - \mathbf{j} x_\eta) \frac{d\eta}{ds} \Big/ \left| \frac{d\mathbf{r}}{ds} \right| \tag{A6}$$

Then the component of $\mathbf{F}$ normal to this ξ-coordinate line is

$$\mathbf{F} \cdot \mathbf{n}^{(\xi)} = (F_1 y_\eta - F_2 x_\eta) \frac{d\eta}{ds} \Big/ \left| \frac{d\mathbf{r}}{ds} \right|$$

so that the flux through this ξ-coordinate line is

$$(\mathbf{F} \cdot \mathbf{n}^{(\xi)}) | d\mathbf{r} | = (F_1 y_\eta - F_2 x_\eta) \, d\eta \tag{A7}$$

Similarly, the unit normal to an η-coordinate line is

$$\mathbf{n}^{(\eta)} = (\mathbf{i}y_\xi - \mathbf{j}x_\xi)\frac{d\xi}{ds}\Big/\left|\frac{d\mathbf{r}}{ds}\right| \tag{A8}$$

so that the flux through this η-coordinate line is

$$(\mathbf{F}\cdot\mathbf{n}^{(\eta)})|\,d\mathbf{r}\,| = (F_1 y_\xi - F_2 x_\xi)\,d\xi \tag{A9}$$

Finally, since $J\,d\xi\,d\eta = dx\,dy$, the integral over the area R' again represents a source within the physical area.

The conservation relation (A5) in the transformed plane thus expresses conservation in the physical plane over the non-rectangular area formed by intersection of the curvilinear ξ and η coordinate lines, while the original relation (A2) expresses conservation over the rectangular area formed by intersection of the straight x and y coordinate lines.

It is emphasized that the transformed relation still expresses conservation over an area in the physical plane. It is clear that the finite difference representation of the conservation relation (A5), properly expressed on a rectangular cell in the transformed plane, centered at (i,j),

$$\begin{aligned}(F_1 y_\eta - F_2 x_\eta)_{i+1/2,j}\,(\Delta\eta) - (F_2 x_\xi - F_1 y_\xi)_{i,j+1/2}\,(-\Delta\xi) + (F_1 y_\eta - F_2 x_\eta)_{i-1/2,j}\,(-\Delta\eta)\\ - (F_2 x_\xi - F_1 y_\xi)_{i,j-1/2}\,(\Delta\xi) = S_{ij}J_{ij}\,\Delta\xi\,\Delta\eta\end{aligned} \tag{A10}$$

does, in view of (A7) and (A9) above, represent conservation over an elemental area in the physical plane. As with the original relation the central difference expression of (A4) in the transformed plane yields the above finite difference expression also. The use of the analytically equivalent relation (A3) would, however, not yield the same finite difference relation or the conservation relation. It is thus possible to express the transformed differential equation in a form which does not preserve the physical conservation relations. However, the point is that the proper expression of the transformed equations will always preserve the physical conservation relations both analytically and on the discrete field.

Now it might be argued that conservation may be more accurately expressed over a rectangular elemental area in the physical field than over an area with curved boundaries, and such may well be the case to some extent in the interior of the field. However, in most partial differential systems the boundary conditions are the dominant influence on the nature of the entire solution. (Recall that all the myriad fluid flow patterns from smooth flow past a flat plate to the oscillatory vortex street are solutions of the same partial differential equations. Just the boundary conditions are different.) For fields with curved boundaries, the rectangular computational cell structure thus is least accurate in the region of strongest influence on the nature of the solution, i.e., near the boundary, and it is this sensitive region that must determine the fineness of mesh. In fact, numerical formulations that are conservative in the interior of the field may not be strictly conservative near curved boundaries because of the use of interpolation and noncoincidence of grid points with the boundary. It is therefore probable that the general coordinate approach can give better overall accuracy than any coordinate system not having coordinate lines coincident with the boundaries.

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